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The nonlinear calculation of Taylor-vortex flow between eccentric rotating cylinders

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This paper is concerned with the nonlinear stability of the flow between two long eccentric rotating cylinders. The problem, which is of interest in lubrication technology, is an extension both of the authors' earlier work on the linear eccentric case and of still earlier work by Davey and others on the nonlinear concentric analysis of Taylor-vortex development. There are four parameters which are assumed small in the analysis; they are the mean clearance ratio, the eccentricity, the amount by which the Taylor number exceeds its critical value; and the Taylor-vortex amplitude. Following the earlier work mentioned above, relationships are specified between these parameters in order to develop a satisfactory perturbation scheme. Thus a non-local solution is obtained to the nonlinear stability problem, in which the whole flow field is taken into account.

Of some importance in the analysis is the fact that it is necessary to allow for the development of a pressure field substantially bigger than that associated with Taylor vortices in the concentric case, owing to the Reynolds lubrication effect in a viscous fluid moving through a converging passage. In order to achieve this mathematically, it is necessary to solve the continuity equation to a higher order than is necessary for the momentum equations.

It is found that the angular position for maximum vortex activity, which is 90° downstream of the maximum gap in the linear case, can taken on any value between 0 and 90°, depending on the value of the supercritical Taylor number. For a particular experiment of Vohr (1968) acceptable agreement is obtained for this angle (50°), though the 'small' parameters are somewhat outside the expected range of perturbation theory. Formulae are obtained for the torque and forces acting on the inner cylinder.

1. Introduction

In two recent papers (DiPrima & Stuart 1972*a*, *b*, hereafter designated as I and II), we have calculated (I) the laminar flow between two long eccentric rotating circular cylinders and (II) the linear stability of the flow against infinitesimal perturbations of Taylor-vortex type. In order to achieve a satisfactory solution of the stability problem, taking properly into account the azimuthal dependence of the basic flow, we devised a perturbation scheme in the eccentricity which allowed for this azimuthal dependence even in the first term of the perturbation series. The method allowed for the fact that the stability depends not only on the local flow at a given azimuthal position, but on the whole flow field; the problem is thus non-local. One dramatic prediction in II is that the position of maximum Taylor-vortex activity is not at the position of widest gap, where the flow is locally most unstable, but is shifted substantially downstream. This result, which arises from the azimuthal dependence of the basic flow and consequent non-local character of the stability problem, is given some support by observations of Vohr (1967, 1968).

The analysis of II was based on the linear stability equations. Here we consider the problem of calculating the Taylor-vortex flow from the nonlinear equations, with three objectives in mind. First, we need to modify the mathematical perturbation scheme to account for the effect on the Taylor-vortex flow of the nonlinearity simultaneously with the non-local flow property. Second, we need to ascertain to what extent the prediction of the linear theory, that the position of maximum Taylor-vortex activity is shifted substantially by non-local effects, is modified by the nonlinearity. Third, and of some importance, we need to calculate the magnitudes of the additional torque and load on the inner cylinder, arising from the presence of the Taylor vortices.

We shall see later that, in order to devise a suitable perturbation scheme, we require an understanding of the lubrication processes by which a large pressure can be developed when fluid flows, or is dragged by boundary movement, from a wider to a narrower part of the channel. This was demonstrated by Reynolds (1886) and is an intrinsic feature of calculations such as that described in I. In the present situation, we anticipate from earlier papers on the concentric problem (Stuart 1958; Davey 1962; Reynolds & Potter 1966; Kirchgässner & Sorger 1969) that an additional component of 'mean' flow is developed by the Reynolds stresses of the Taylor vortices, where 'mean' is used in the sense of an average along the axis of the cylinders. Since, in the case of eccentric cylinders, the gap between them varies around the annulus, we have the likelihood that an additional large pressure will be developed by the Reynolds lubrication effect, in association with the modified mean flow due to the Taylor vortices. This effect must be allowed for in the perturbation expansion of the pressure in the nonlinear calculation.

We now proceed as follows. The basic nonlinear equations are given in §2, essentially following II, but with a few differences in notation. Then the perturbation expansion (in terms of the eccentricity ϵ and the curvature parameter δ) is proposed and explained in §3, especially with reference to the 'lubrication' effect, and is followed by application to the nonlinear equations. The resulting sets of equations, ordered in $\epsilon^{\frac{1}{2}}$, are solved sequentially in §4. A nonlinear generalization of the linear amplitude equation of II is deduced by a method rather like that of multiple scales. A discussion of the flow and pressure fields, including the differences from the linear theory of II, is given in §5. The torque and load on the inner cylinder are calculated in §6, and a general discussion follows in §7.

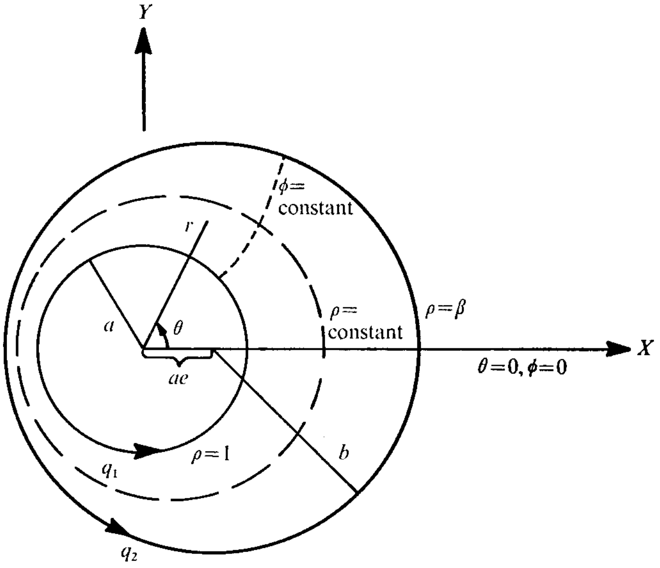

FIGURE 1. Geometry and co-ordinate systems.

A preliminary and summarized account of this work was given at the Lubrication Symposium organized by the A.S.M.E. at Northwestern University in June 1973 (DiPrima & Stuart 1974).

2. The basic equations

Following II we consider two cylinders of radii a and b and linear speeds q_1 and q_2 measured anticlockwise. The centres of the cylinders are set a distance ae apart, where

$$e = \epsilon\delta, \quad \delta = (b-a)/a, \quad 0 \leqslant \epsilon < 1. \tag{2.1}$$

Here ϵ is the eccentricity and δ is the mean clearance ratio. Instead of (r, θ) polar co-ordinates, we use Wood's (1957) modified bipolar co-ordinates (ρ, ϕ) (see equation (2.3) of II), where the curves ρ = constant are circles, the inner and outer cylinders being given by $\rho = 1$ and $\rho = \beta$. Moreover, the orthogonal curves ϕ = constant are such that $\phi = 0$ coincides with $\theta = 0$ (figure 1). The parameter β is defined by equations (2.4) and (2.5) of II. The basic laminar flow is given in I for arbitrary ϵ $(0 \leqslant \epsilon < 1)$ in the form of a power-series expansion for small values of $\alpha = \beta - 1 = \delta(1-\epsilon^2)^{\frac{1}{2}}[1+O(\delta)]$ and of the modified Reynolds number $R_m = (q_1 a/\nu)\alpha^2$, where ν is the kinematic viscosity.

The linear stability problem in II is solved for both α and ϵ tending to zero, but with the Taylor number T, defined in (2.3), held fixed or allowed to vary only in a limited range and with the constraint

$$\alpha^{\frac{1}{2}} = k\epsilon(2c)^{\frac{1}{2}}, \tag{2.2}$$

where k is a parameter held fixed as $\epsilon \to 0$. For a given experiment, α, ϵ and c are known, and k takes on the value which satisfies (2.2). The following definitions are relevant:

$$\left.\begin{aligned} c &= 2(q_1-q_2)/(q_1+q_2), \quad R_a = \tfrac{1}{2}(1+q_2/q_1)\,R_m = \tfrac{1}{2}(q_1+q_2)\,a\alpha^2/\nu, \\ T &= (q_1 a/\nu)^2\alpha^3(1-q_2^2/q_1^2) = 2cR_a^2/\alpha. \end{aligned}\right\} \tag{2.3}$$

The Taylor number takes on a conventional definition for the concentric problem, for which $\epsilon = 0$ and, therefore, $\alpha = \delta$. The last of (2.3) is especially notable, since it implies that $R_a \sim \alpha^{\frac{1}{2}} \sim \epsilon$ as $\epsilon \to 0$ with the Taylor number T held fixed. The reasons for considering δ and, therefore, α to be small are twofold: on the one hand, several experiments have been done with δ of order 0·1 or even as small as 0·01; on the other hand, small values of δ are of prime importance in lubrication technology, which is one of our concerns. The relation (2.2) follows from the desire to make the terms $R_a\partial/\partial\phi$ and $\epsilon\cos\phi$ remain in balance. (This is true for the actual perturbation equations, but can be seen more clearly in the model equation (3.1).) Although this is the method we follow in this paper, because of its mathematical interest and experimental relevance, there are other possible lines of attack: if ϵ were small, but δ were not, a straightforward expansion in ϵ could be used; if δ were small, but ϵ were not, a variation on the WKB method would be appropriate; in the general case, a Galerkin or other numerical approach could be employed.

It is proposed in the present paper to solve the nonlinear problem, subject to (2.2) and to the constraint on T mentioned above, in terms of a series involving some power of ϵ. To this end we shall need to know the basic laminar flow as a power series in ϵ. From I and II we have the following approximations for the (ρ, ϕ) components of velocity, which are valid as $\epsilon \to 0$ with T fixed by the requirement $R_a \sim \epsilon$:

$$u_\rho(x,\phi) = \tfrac{1}{2}\alpha\epsilon(q_1+q_2)\,U(x,\phi), \quad u_\phi(x,\phi) = \tfrac{1}{2}(q_1+q_2)\,V(x,\phi), \tag{2.4}$$

where

$$U = 2(x^2-\tfrac{1}{4})\,(x-\tfrac{1}{4}c)\sin\phi + O(\epsilon), \tag{2.5}$$

$$V = V_0(x) + \epsilon V_1(x,\phi) + \epsilon^2[V_{20}(x) + kT^{\frac{1}{2}}V_{21}(x,\phi) + k^2c^2V_{22}(x)] + O(\epsilon^3), \tag{2.6}$$

$$\left.\begin{aligned} &V_0 = 1-cx, \quad V_1 = 6(x^2-\tfrac{1}{4})\cos\phi, \quad V_{20} = 3(x^2-\tfrac{1}{4}), \\ &V_{21} = (x^2-\tfrac{1}{4})\,[\tfrac{1}{2}(1-\tfrac{1}{12}c^2)\,(\tfrac{1}{20}-x^2) - \tfrac{1}{5}cx(\tfrac{7}{12}-x^2)]\sin\phi, \\ &V_{22} = x^2-\tfrac{1}{4} \end{aligned}\right\} \tag{2.7}$$

and

$$\rho - 1 = \alpha(x+\tfrac{1}{2}). \tag{2.8}$$

Knowledge of the concentric nonlinear problem (Stuart 1958; Davey 1962; II) then suggests that the perturbed velocity and pressure fields can be written as

$$u_\rho(x,\phi,\xi,\tau) = \tfrac{1}{2}\alpha\epsilon(q_1+q_2)\,U(x,\phi) + (\nu/a\alpha)\,u(x,\phi,\xi,\tau), \tag{2.9}$$

$$u_\phi(x,\phi,\xi,\tau) = \tfrac{1}{2}(q_1+q_2)\,V(x,\phi) + (q_1-q_2)\,v(x,\phi,\xi,\tau), \tag{2.10}$$

$$u_\xi(x,\phi,\xi,\tau) = (\nu/a\alpha)\,w(x,\phi,\xi,\tau), \tag{2.11}$$

$$p'(x,\phi,\xi,\tau) = (\nu q_1/a\alpha^2)\,P(x,\phi) + (\nu^2/a^2\alpha^2)\,p(x,\phi,\xi,\tau), \tag{2.12}$$

where $a\alpha\xi$ is the axial co-ordinate, u_ξ is the axial velocity, $\tau = \nu t/a^2\alpha^2$ is the non-dimensional form of the time t and p' is the kinematic pressure. The above scalings reflect the well-known fact that, in the concentric problem, the radial and axial perturbation velocity components are smaller than the azimuthal component by a Reynolds number factor (Stuart 1958). With regard to the pressure scaling, however, a cautious attitude is necessary, since we have not allowed for a large pressure development due to Reynolds' lubrication effect. We return to this point in § 3.

Later the flow will be assumed to be steady, since our main mathematical concern is to calculate the steady Taylor vortex. However, it may be helpful for future work to retain $\partial/\partial\tau$ in the general formulation. Apart from the replacing of t by τ, and the exchange of p and p', the above formulae are identical with equations (3.5)–(3.8) of II.

The equations of motion and the continuity equation in the modified bipolar co-ordinate system are

$$\begin{aligned}\frac{\partial u}{\partial\tau}+\epsilon R_a J^{\frac{1}{2}}\left[U\frac{\partial u}{\partial x}+u\frac{\partial U}{\partial x}\right]+\frac{R_a J^{\frac{1}{2}}}{\rho}\left[V\frac{\partial u}{\partial\phi}+\epsilon c R_a v\frac{\partial U}{\partial\phi}\right]&\\ -R_a\frac{\partial}{\partial\phi}\left(\frac{J^{\frac{1}{2}}}{\rho}\right)(Vu+\epsilon cR_a vU)+\rho\frac{\partial}{\partial\rho}\left(\frac{J^{\frac{1}{2}}}{\rho}\right)TVv&\\ +\left[J^{\frac{1}{2}}u\frac{\partial u}{\partial x}+\frac{J^{\frac{1}{2}}}{\rho}cR_a v\frac{\partial u}{\partial\phi}+w\frac{\partial u}{\partial\xi}-\frac{1}{\rho}\frac{\partial}{\partial\phi}(J^{\frac{1}{2}})cR_a uv+\tfrac{1}{2}cT\rho\frac{\partial}{\partial\rho}\left(\frac{J^{\frac{1}{2}}}{\rho}\right)v^2\right]&\\ =-J^{\frac{1}{2}}\frac{\partial p}{\partial x}+\left(JL^2+\frac{\partial^2}{\partial\xi^2}\right)u-\alpha^2 Hu&\\ +cR_a\rho\left\{\alpha\frac{\partial}{\partial\rho}\left(\frac{J}{\rho^2}\right)\frac{\partial v}{\partial\phi}-\frac{\partial}{\partial\phi}\left(\frac{J}{\rho^2}\right)\frac{\partial v}{\partial x}\right\},&\end{aligned}\tag{2.13}$$

$$\begin{aligned}\frac{\partial v}{\partial\tau}+J^{\frac{1}{2}}\left[\frac{1}{c}u\frac{\partial V}{\partial x}+\epsilon R_a U\frac{\partial v}{\partial x}\right]+\frac{1}{\rho}J^{\frac{1}{2}}R_a\left[V\frac{\partial v}{\partial\phi}+v\frac{\partial V}{\partial\phi}\right]&\\ -\alpha\rho\frac{\partial}{\partial\rho}\left(\frac{J^{\frac{1}{2}}}{\rho}\right)\left[\frac{1}{c}Vu+\epsilon R_a Uv\right]+2\frac{\alpha^2\epsilon}{c}\frac{\partial}{\partial\phi}\left(\frac{J^{\frac{1}{2}}}{\rho}\right)uU&\\ +\left[J^{\frac{1}{2}}u\frac{\partial v}{\partial x}+\frac{J^{\frac{1}{2}}}{\rho}cR_a v\frac{\partial v}{\partial\phi}+w\frac{\partial v}{\partial\xi}-\rho\frac{\partial}{\partial\rho}\left(\frac{J^{\frac{1}{2}}}{\rho}\right)\alpha uv+\frac{1}{\rho}\frac{\partial}{\partial\phi}(J^{\frac{1}{2}})\frac{\alpha^2}{cR_a}u^2\right]&\\ =-\frac{J^{\frac{1}{2}}}{\rho}\frac{\alpha^2}{cR_a}\frac{\partial p}{\partial\phi}+\left(JL^2+\frac{\partial^2}{\partial\xi^2}\right)v-\alpha^2Hv&\\ +\frac{\rho\alpha^2}{cR_a}\left\{-\alpha\frac{\partial}{\partial\rho}\left(\frac{J}{\rho^2}\right)\frac{\partial u}{\partial\phi}+\frac{\partial}{\partial\phi}\left(\frac{J}{\rho^2}\right)\frac{\partial u}{\partial x}\right\},&\end{aligned}\tag{2.14}$$

$$\begin{aligned}\frac{\partial w}{\partial\tau}+\epsilon J^{\frac{1}{2}}R_a U\frac{\partial w}{\partial x}+\frac{1}{\rho}J^{\frac{1}{2}}R_aV\frac{\partial w}{\partial\phi}+\left[J^{\frac{1}{2}}u\frac{\partial w}{\partial x}+\frac{J^{\frac{1}{2}}}{\rho}cR_a v\frac{\partial w}{\partial\phi}+w\frac{\partial w}{\partial\xi}\right]&\\ =-\partial p/\partial\xi+(JL^2+\partial/\partial\xi^2)w,&\end{aligned}\tag{2.15}$$

$$J^{\frac{1}{2}}\frac{\partial u}{\partial x}-\alpha\rho\frac{\partial}{\partial\rho}\left(\frac{J^{\frac{1}{2}}}{\rho}\right)u+cR_a\left[\frac{J^{\frac{1}{2}}}{\rho}\frac{\partial v}{\partial\phi}-v\frac{\partial}{\partial\phi}\left(\frac{J^{\frac{1}{2}}}{\rho}\right)\right]+\frac{\partial w}{\partial\xi}=0,\tag{2.16}$$

where J is the Jacobian of the conformal transformation defined in I and II (§ 2), and

$$L^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\alpha}{\rho}\frac{\partial}{\partial x}+\left(\frac{\alpha}{\rho}\right)^2\frac{\partial^2}{\partial\phi^2},\tag{2.17}$$

$$H\equiv\rho J^{\frac{1}{2}}\left\{\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right\}\left(\frac{J^{\frac{1}{2}}}{\rho}\right).\tag{2.18}$$

In view of the complex nature of the perturbation equations (2.13)–(2.16), it is impossible in a reasonable amount of space to give all the details which may be of interest. There are, however, certain important features to which we would like to draw attention: (i) as can be seen from the series expansions of the basic flow (2.5)–(2.7) and of the Jacobian (3.9), terms proportional to $\epsilon \cos\phi$ and $\epsilon^2 \sin\phi$ appear in (2.13)–(2.16); (ii) when it occurs, the derivative $\partial/\partial\phi$, which represents convective effects, is usually multiplied by the parameter R_a, which is $O(\epsilon)$ according to (2.2) and (2.3).

3. The perturbation expansion

In II it was thought to be helpful to explain the method to be used by reference to a simpler *model* equation, before proceeding to the application of the method to the true (and involved) equations of the stability problem. A similar procedure may be helpful here, in explaining the method to be used to solve the nonlinear equations (2.13)–(2.16). For purposes of explanation only, therefore, let us consider the steady equation

$$\nabla^2\left(\nabla^2 - R_a\frac{\partial}{\partial\phi}\right)^2 v - T(1+\epsilon\cos\phi)\frac{\partial^2 v}{\partial\xi^2} = \frac{\partial v^2}{\partial x}, \tag{3.1}$$

where

$$\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial\xi^2. \tag{3.2}$$

The essential differences between this equation and equation (4.1) of II lie in the inclusion of a right-hand side which is quadratic in v, and in the replacement of $-\lambda^2 v$ by $\partial^2 v/\partial\xi^2$.

According to (2.2) and (2.3) we have $R_a = (T\alpha/2c)^{\frac{1}{2}} = T^{\frac{1}{2}}k\epsilon$. Thus the coefficients in (3.1) are functions of ϵ and T, with k fixed. In II the linear form of (3.1) is solved by an expansion in ϵ of the following form:

$$v = A[v^{(0)}(x,\phi) + \epsilon v^{(1)}(x,\phi) + O(\epsilon^2)]\cos\lambda\xi, \tag{3.3}$$

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \dots, \tag{3.4}$$

where $v^{(0)}$ and $v^{(1)}$ are functions to be determined, T_0, T_1 and T_2 are numbers to be determined by integrability conditions, and A is an arbitrary amplitude of the linear eigensolution. We note, however, a difference between the role of T_1 here and in the linear problem of II. In the latter case T_1 is zero for a neutral perturbation; this corresponds to the Taylor number taking on its critical value, as a function of ϵ. In the nonlinear case, on the other hand, we shall find that T_1 can be chosen to specify the amount, of order ϵ, by which the Taylor number exceeds its critical value.

In solving the nonlinear problem (3.1) with (3.2), it is necessary to determine A and to allow for the generation of harmonic and mean (in ξ) effects by the nonlinear term. Because of the quadratic character of the nonlinear term, it is clear that, if the fundamental ($\cos\lambda\xi$) has small magnitude A, the mean and first harmonic ($\cos 2\lambda\xi$) terms appear with magnitude A^2, and a correction to the fundamental and the second harmonic ($\cos 3\lambda\xi$) with magnitude A^3. For the linear problem, the term $v^{(0)}$ of (3.3) is known (II) to be of the form $B(\phi)$ times a simple eigenfunction, namely that for the concentric case; the amplitude function $B(\phi)$ is determined with $v^{(1)}$ at order ϵA. This suggests that, if we wish to allow for this correction due to eccentricity *simultaneously* with the nonlinear effect on the fundamental, which is of order A^3, we should make the identification $\epsilon A = A^3$: this yields $A = \epsilon^{\frac{1}{2}}$. Allowing for this, and for mean and harmonic effects (of order $A^2 = \epsilon$), we replace (3.3) by

$$v = \epsilon^{\frac{1}{2}}[v^{(0)}(x,\phi,\xi) + \epsilon^{\frac{1}{2}}v^{(1)}(x,\phi,\xi) + \epsilon v^{(2)}(x,\phi,\xi)] + O(\epsilon^2), \tag{3.5}$$

with (3.4) remaining unchanged. Since the amplitude of the disturbance is now proportional to $\epsilon^{\frac{1}{2}}$, it is necessary to include all powers of $\epsilon^{\frac{1}{2}}$ in the expansion of the velocity and pressure fields, as is well known in nonlinear stability theory (Davey 1962). On the other hand, odd powers of $\epsilon^{\frac{1}{2}}$ in the expansion (3.4) for T can be shown to be zero, and for convenience have been omitted.

Our plan now is to apply (2.2), (3.4) and expansions like (3.5) to the true equations (2.13)–(2.15) of our problem in the steady case, in order to calculate the equilibrium Taylor-vortex flow and associated properties. Thus for the velocity field we write

$$u = \epsilon^{\frac{1}{2}}[u_0(x,\phi,\xi) + \epsilon^{\frac{1}{2}}u_1(x,\phi,\xi) + \epsilon u_2(x,\phi,\xi) + \epsilon^{\frac{3}{2}}u_3(x,\phi,\xi) + \epsilon^2 u_4(x,\phi,\xi)] + O(\epsilon^3), \tag{3.6}$$

$$v = \epsilon^{\frac{1}{2}}[v_0(x,\phi,\xi) + \epsilon^{\frac{1}{2}}v_1(x,\phi,\xi) + \epsilon v_2(x,\phi,\xi) + \epsilon^{\frac{3}{2}}v_3(x,\phi,\xi) + \epsilon^2 v_4(x,\phi,\xi)] + O(\epsilon^3), \tag{3.7}$$

$$w = \epsilon^{\frac{1}{2}}[w_0(x,\phi,\xi) + \epsilon^{\frac{1}{2}}w_1(x,\phi,\xi) + \epsilon w_2(x,\phi,\xi) + \epsilon^{\frac{3}{2}}w_3(x,\phi,\xi) + \epsilon^2 w_4(x,\phi,\xi)] + O(\epsilon^3). \tag{3.8}$$

The model equation, however, does not give immediate help with the choice of the expansion for the pressure field. But we can pinpoint a possibly related difficulty, which in itself gives a clue to the choice of pressure expansion, by reference to the continuity equation (2.16). Substituting (3.6)–(3.8) in the latter, using (2.2), (2.3), (2.8) and (3.4), and making use of an expansion of J, namely

$$J = 1 - 2\epsilon\cos\phi + \epsilon^2(1+\cos^2\phi) + O(\epsilon^2), \tag{3.9}$$

we obtain

$$\partial u_0/\partial x + \partial w_0/\partial\xi = 0 \quad \text{at} \quad O(\epsilon^{\frac{1}{2}}), \tag{3.10}$$

$$\partial u_1/\partial x + \partial w_1/\partial\xi = 0 \quad \text{at} \quad O(\epsilon), \tag{3.11}$$

$$\frac{\partial u_2}{\partial x} - \frac{\partial u_0}{\partial x}\cos\phi + ckT_0^{\frac{1}{2}}\frac{\partial v_0}{\partial\phi} + \frac{\partial w_2}{\partial\xi} = 0 \quad \text{at} \quad O(e^{\frac{3}{2}}), \tag{3.12}$$

$$\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial x}\cos\phi + ckT_0^{\frac{1}{2}}\frac{\partial v_1}{\partial\phi} + \frac{\partial w_3}{\partial\xi} = 0 \quad \text{at} \quad O(\epsilon^2). \tag{3.13}$$

These continuity conditions have to be solved in succession, in association with corresponding momentum equations. We are particularly interested in analysing the conditions that (3.9)–(3.13) impose on the mean flow, which is that part independent of ξ in a Fourier analysis. At $O(\epsilon^{\frac{1}{2}})$ the equations are essentially those of linear stability theory for concentric rotating cylinders; in their solution, however, an amplitude function $B(\phi)$ must be introduced to multiply the cos $\lambda\xi$ fundamental, as indicated in II. The $O(\epsilon)$ equations relate to the mean field, which is independent of ξ, and to the first harmonic, which is of the form $\cos 2\lambda\xi$. Moreover if u_{10} denotes the term in u which is independent of ξ, then (3.11) yields $\partial u_{10}/\partial x = 0$, which, with the no-slip boundary conditions, implies that $u_{10} \equiv 0$. Then at $O(\epsilon^{\frac{3}{2}})$ we have a correction to the fundamental, together with a second-harmonic term, which is proportional to $\cos 3\lambda\xi$. It is at $O(\epsilon^2)$ that the mean field returns; and we see from (3.13) that, for terms independent of ξ, we have

$$\partial u_{30}/\partial x + ckT_0^{\frac{1}{2}}\,\partial v_{10}/\partial\phi = 0, \tag{3.14}$$

where v_{10} denotes the mean part of v_1 and u_{30} the mean part of u_3.

On the assumption that v_{10} has been calculated at $O(\epsilon)$, equation (3.14) can be integrated to yield

$$u_{30}(x,\phi) = -ckT_0^{\frac{1}{2}}\int_{-\frac{1}{2}}^{x}\frac{\partial v_{10}}{\partial\phi}\,dx, \tag{3.15}$$

where the no-slip boundary condition has been applied at $x = -\frac{1}{2}$. However, we have not satisfied the corresponding condition at $x = +\frac{1}{2}$; furthermore, if the assumption that v_{10} is completely determinate at $O(\epsilon)$ is valid, we have no chance of satisfying that condition.

A resolution of this dilemma can be seen from a consideration of lubrication theory: it is known from that theory that the requirement of satisfying continuity of mass flux in a non-parallel channel (or, in other words, the no-slip boundary condition) provides for the development of an additional, and usually strong, pressure field. Indeed Reynolds' (1886) lubrication equation for the pressure can be derived from just that requirement; this is well known, but one reference source is I.

In the present situation we wish somehow to allow for an arbitrary parameter in v_{10}, dependent on ϕ, which can be used to satisfy the additonal condition on u_{30}. Lubrication theory is, in fact, suggesting to us that an additional pressure should be introduced of such scale as to modify the momentum equation at $O(\epsilon)$ and thus v_{10}. Moreover, since the pressure gradient $\partial p/\partial\phi$ in (2.14) is multiplied by $\alpha^2 R_a^{-1} c^{-1} = 4ck^3\epsilon^3 T^{-\frac{1}{2}} = O(\epsilon^3)$, as can be seen from (2.2) and (2.3), it is clear that p has to be large, in fact $O(\epsilon^{-2})$ in our scaling, in order to be effective at $O(\epsilon)$. The arguments suggest that p should be expanded as

$$\begin{aligned} p = \frac{T^{\frac{1}{2}}\epsilon^{-2}}{4ck^3}&\left[\int_0^{\phi} q_0(\phi)\,d\phi + \epsilon\int_0^{\phi} q_1(\phi)\,d\phi + O(\epsilon^2)\right] \\ &+ \epsilon^{\frac{1}{2}}[p_0(x,\phi,\xi) + \epsilon^{\frac{1}{2}}p_1(x,\phi,\xi) + \epsilon p_2(x,\phi,\xi) + \epsilon^{\frac{3}{2}}p_3(x,\phi,\xi) \\ &+ \epsilon^2 p_4(x,\phi,\xi) + O(\epsilon^{\frac{5}{2}})] + \text{constant}. \end{aligned} \tag{3.16}$$

Those terms which are within square brackets and multiplied by $\epsilon^{\frac{1}{2}}$ form the basic x- and ξ-dependent pressure field brought about by the usual Taylor-vortex effect (Davey 1962). On the other hand, the other group of terms, those within square brackets and multiplied by ϵ^{-2}, forms the 'lubrication' or Reynolds pressure field. It can be shown that this part of the field must be independent of x, in order for the expansions (3.6)–(3.8) and (3.16) to be consistent with (2.13)–(2.16). Moreover, we note that the lack of dependence of the pressure on x is consistent with Reynolds' lubrication theory. The expansion parameter is ϵ, and not $\epsilon^{\frac{1}{2}}$, because the arbitrary functions $q_0(\phi)$, $q_1(\phi)$, ..., appear only in the mean equations at $O(\epsilon)$, $O(\epsilon^2)$ and so on. Although we have chosen to represent (3.16) as two series added together, the two parts do interact and can be conceived as a single Laurent series in $\epsilon^{\frac{1}{2}}$. A few further remarks about the pressure scaling may be in order. The assumed scaling (2.12) represents the effect of the Taylor vortices and the generation of a modified mean flow and pressure by the associated Reynolds stress. This is an effect present in the concentric problem. When the cylinders are placed eccentrically, however, the 'lubrication' effect of the forcing of fluid through a variable passage produces a much larger pressure. From (2.12) and (3.16) the kinematic pressure is

$$(\nu^2/a^2\alpha^2)(T^{\frac{1}{2}}\epsilon^{-2}/4ck^3) = (\epsilon\nu q_1/a\alpha^2)(1-q^2/q_1).$$

Except for having a factor $1-q_2/q_1$ instead of $1+q_2/q_1$, this is consistent with classical lubrication theory. The effect of eccentricity (at first order) in the basic flow is to introduce a plane Poiseuille component $O(\epsilon)$ and a pressure $O(\epsilon/\delta^2)$ as shown in equation (90) of I. Here, in the perturbation, the nonlinear terms produce a mean flow $O(\epsilon)$, including a plane Poiseuille term, together with a lubrication pressure also $O(\epsilon/\delta^2) = O(\epsilon/\alpha^2)$, which is the $O(T^{\frac{1}{2}}\epsilon^{-2})$ term of (3.16). We shall see later that, as the limit of the concentric case is approached, this lubrication pressure field disappears.

We are in a position now to write down the sets of steady equations that we wish to solve at successive orders. These equations are derivable from (2.13)–(2.16), and with use of (2.2), (2.3), (2.5)–(2.8), (3.4), (3.6)–(3.9) and (3.16) together with

$$M \equiv \partial^2/\partial x^2 + \partial^2/\partial\xi^2, \tag{3.17}$$

yield the following:

$$-T_0V_0v_0 = -\partial p_0/\partial x + Mu_0, \tag{3.18}$$

$$-u_0 = Mv_0, \tag{3.19}$$

$$0 = -\partial p_0/\partial\xi + Mw_0 \tag{3.20}$$

together with (3.10) at $O(\epsilon^{\frac{1}{2}})$;

$$-T_0V_0v_1 + u_0\frac{\partial u_0}{\partial x} + w_0\frac{\partial u_0}{\partial\xi} - \tfrac{1}{2}cT_0v_0^2 = -\frac{\partial p_1}{\partial x} + Mu_1, \tag{3.21}$$

$$-u_1 + u_0\,\partial v_0/\partial x + w_0\,\partial v_0/\partial\xi = -q_0(\phi) + Mv_1, \tag{3.22}$$

$$u_0\frac{\partial w_0}{\partial x} + w_0\frac{\partial w_0}{\partial\xi} = -\frac{\partial p_1}{\partial\xi} + Mw_1 \tag{3.23}$$

together with (3.11) at $O(\epsilon)$;

$$\begin{aligned} &-T_0V_0v_2 + kT_0^{\frac{1}{2}}V_0\frac{\partial u_0}{\partial\phi} - T_0V_1v_0 - T_1V_0v_0 \\ &+ u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x} + w_0\frac{\partial u_1}{\partial\xi} + w_1\frac{\partial u_0}{\partial\xi} - cT_0v_0v_1 \\ &\qquad = -\frac{\partial p_2}{\partial x} + \frac{\partial p_0}{\partial x}\cos\phi + Mu_2 - 2\frac{\partial^2 u_0}{\partial x^2}\cos\phi, \end{aligned} \tag{3.24}$$

$$-u_2+u_0\cos\phi+kT_0^{\frac{1}{2}}V_0\frac{\partial v_0}{\partial\phi}+\frac{1}{c}\frac{\partial V_1}{\partial x}u_0$$

$$+u_0\frac{\partial v_1}{\partial x}+u_1\frac{\partial v_0}{\partial x}+w_0\frac{\partial v_1}{\partial\xi}+w_1\frac{\partial v_0}{\partial\xi}=Mv_2-2\frac{\partial^2 v_0}{\partial x^2}\cos\phi, \quad (3.25)$$

$$kT_0^{1}V_0\frac{\partial w_0}{\partial\phi}+u_0\frac{\partial w_1}{\partial x}+u_1\frac{\partial w_0}{\partial x}+w_0\frac{\partial w_1}{\partial\xi}+w_1\frac{\partial w_0}{\partial\xi}=-\frac{\partial p_2}{\partial\xi}+Mw_2-2\frac{\partial^2 w_0}{\partial x^2}\cos\phi \quad (3.26)$$

together with (3.12) at $O(\epsilon^{\frac{3}{2}})$.

This is the order to which we shall take the calculations in the present paper. However, we shall need to consider in addition the mean continuity equation at $O(\epsilon^2)$, as given by (3.14) and its solution (3.15). The pressure gradient $q_0(\phi)$, which contributes a 'plane Poiseuille' or 'lubrication' effect in the $O(\epsilon)$ azimuthal momentum equation (3.22), can be determined by satisfaction of the outstanding boundary condition on u_{30} in (3.15).

4. Solution of the equations

Order $\epsilon^{\frac{1}{2}}$

Following II (§5), we find that the system of equations (3.18)–(3.20) and (3.10), subject to $u_0 = v_0 = w_0 = 0$ at $x = \pm\frac{1}{2}$, determines an eigenvalue problem for T_0. The solution is

$$\left.\begin{aligned} u_0 &= -B(\phi)f_0(x)\cos\lambda\xi, \quad v_0 = B(\phi)\,g_0(x)\cos\lambda\xi, \\ w_0 &= \lambda^{-1}B(\phi)\,Df_0\sin\lambda\xi, \quad p_0 = -\lambda^{-2}B(\phi)\,(D^2-\lambda^2)\,Df_0\cos\lambda\xi, \end{aligned}\right\} \quad (4.1)$$

where λ denotes the axial wavenumber, D denotes d/dx, $B(\phi)$ is an amplitude function to be determined (as in II), and f_0 and g_0 are defined by equations (5.4)–(5.6) of II with $\sigma = 0$. This is the concentric stability problem, with a critical Taylor number of $T_0 = 1695$ at $\lambda = 3{\cdot}127$; these are the eigenvalues we shall use in this paper.

Order ϵ

From (4.1) the nonlinear terms of (3.21)–(3.23) can be calculated, and yield both mean and first-harmonic contributions. The solution of these equations, together with (3.11), has the form

$$\left.\begin{aligned} u_1 &= -B^2(\phi)f_{12}(x)\cos 2\lambda\xi, \quad v_1 = v_{10}(x,\phi)+B^2(\phi)\,g_{12}(x)\cos 2\lambda\xi, \\ w_1 &= (2\lambda)^{-1}B^2(\phi)\,Df_{12}(x)\sin 2\lambda\xi, \quad p_1 = p_{10}(x,\phi)+B^2(\phi)\,l_{12}(x)\cos 2\lambda\xi. \end{aligned}\right\} \quad (4.2)$$

In these formulae f_{12} and g_{12} are given by the non-homogeneous ordinary differential system

$$\left.\begin{aligned} f_{12}-M_2 g_{12} &= \tfrac{1}{2}(f_0 Dg_0-g_0 Df_0), \\ M_2^2 f_{12}+4\lambda^2\,T_0 V_0 g_{12} &= -f_0 D^3 f_0+(Df_0)\,D^2 f_0-c\lambda^2 T_0 g_0^2, \\ f_{12} = Df_{12} = g_{12} &= 0, \quad x = \pm\tfrac{1}{2}, \end{aligned}\right\} \quad (4.3)$$

where $$M_n = D^2-n^2\lambda^2. \quad (4.4)$$

The harmonic pressure function $l_{12}(x)$ is given by

$$l_{12} = (2\lambda)^{-2}[-M_2 Df_{12}+(Df_0)^2-f_0 D^2 f_0]. \quad (4.5)$$

We turn now to the calculation of the mean field, that is $v_{10}(x,\phi)$, $q_0(\phi)$ and $p_{10}(x,\phi)$. From (3.22) we obtain

$$-\tfrac{1}{2}B^2(\phi)\,D(f_0g_0) = -q_0(\phi)+D^2v_{10}. \qquad (4.6)$$

This ordinary differential equation can easily be integrated for v_{10}, subject to $v_{10}=0$ at $x=\pm\frac{1}{2}$, since $B(\phi)$ and $q_0(\phi)$ are independent of x. Substituting v_{10} then in equation (3.15) for u_{30}, and satisfying the boundary condition $u_{30}=0$ at $x=\pm\frac{1}{2}$, we obtain an expression for $q_0(\phi)$, namely

$$q_0(\phi) = -6B^2(\phi)\,Q_0+\sigma_0, \qquad (4.7)$$

where σ_0 is a constant still to be determined and Q_0 is a constant defined by

$$F_0(x) = \int_{-\frac{1}{2}}^{x} f_0g_0\,dx-(x+\tfrac{1}{2})\int_{-\frac{1}{2}}^{\frac{1}{2}} f_0g_0\,dx \qquad (4.8)$$

and

$$Q_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} F_0(x)\,dx. \qquad (4.9)$$

The constant σ_0 is now chosen to satisfy the requirement that the pressure (3.16) must be periodic in ϕ with period 2π, so that

$$0 = \int_0^{2\pi} q_0(\phi)\,d\phi = \int_0^{2\pi} (\sigma_0-6B^2(\phi)\,Q_0)\,d\phi. \qquad (4.10)$$

This constraint becomes useful once we have obtained an equation for $B(\phi)$, which we find at $O(\epsilon^{\frac{3}{2}})$.

From (4.6) and (4.7) we now have

$$v_{10}(x,\phi) = B^2(\phi)\,g_{10}(x)+\tfrac{1}{2}\sigma_0(x^2-\tfrac{1}{4}), \qquad (4.11a)$$

where

$$g_{10} \equiv -\tfrac{1}{2}F_0(x)-3Q_0(x^2-\tfrac{1}{4}). \qquad (4.11b)$$

Thus

$$v_{10}(x,\phi) = -\tfrac{1}{2}B^2(\phi)\,F_0(x)+[\tfrac{1}{2}\sigma_0-3Q_0B^2(\phi)]\,(x^2-\tfrac{1}{4}). \qquad (4.12)$$

Moreover (3.21) yields

$$p_{10}(x,\phi) = B^2(\phi)\,l_{10}(x)+\sigma_0 l_{100}(x), \qquad (4.13)$$

where

$$l_{10} = \int_{-\frac{1}{2}}^{x} [T_0V_0(x)\,g_{10}-f_0Df_0+\tfrac{1}{4}cT_0g_0^2]\,dx+\text{constant} \qquad (4.14)$$

and

$$l_{100} = \frac{T_0}{2}\int_{-\frac{1}{2}}^{x} V_0(x)\,(x^2-\tfrac{1}{4})\,dx+\text{constant}. \qquad (4.15)$$

It is of interest to note that the term $F_0(x)$ in (4.12) represents the contribution to v_{10} from the Reynolds stress of the Taylor vortex, while the term proportional to $x^2-\frac{1}{4}$ is the plane-Poiseuille-flow contribution, arising from the Reynolds or lubrication effect.

Order $\epsilon^{\frac{3}{2}}$

From (4.1), (4.2) and (4.11) we can calculate the nonlinear terms which arise in (3.24)–(3.26). It is clear, from the form of the interactions between $O(\epsilon^{\frac{1}{2}})$ and $O(\epsilon)$ terms, that the nonlinearities will generate both the fundamental wavenumber λ and the second harmonic 3λ. Thus it is convenient to separate these different harmonics by writing

$$\left.\begin{aligned} u_2 &= -u_{21}(x,\phi)\cos\lambda\xi-u_{23}(x,\phi)\cos 3\lambda\xi,\\ v_2 &= v_{21}(x,\phi)\cos\lambda\xi+v_{23}(x,\phi)\cos 3\lambda\xi, \end{aligned}\right\} \qquad (4.16)$$

with similar expressions for w_2 and p_2.

For the fundamental component, (3.25) then yields

$$-M_1 v_{21}+u_{21} = [B(\phi)\cos\phi]\,G_{111}(x)+kT_0^{\frac{1}{2}}(dB/d\phi)\,G_{112}(x)+B^3(\phi)\,G'_{114}(x) \\ +\sigma_0 B(\phi)\,xf_0(x). \quad (4.17)$$

Moreover, by eliminating p_2 between (3.24) and (3.26) and by using (3.12), we have

$$M_1^2 u_{21}+\lambda^2 T_0 V_0 v_{21} = [B(\phi)\cos\phi]\,F_{111}(x)+kT_0^{\frac{1}{2}}(dB/d\phi)\,F_{112}(x) \\ +B^3(\phi)\,F'_{114}(x)+T_1 B(\phi)\,F_{113}(x)-\tfrac{1}{2}\sigma_0\lambda^2 cT_0 g_0(x)\,(x^2-\tfrac{1}{4})\,B(\phi). \quad (4.18)$$

We note that $dB/d\phi$ appears in (4.17) and (4.18), where the unprimed F and G functions are given by equations (5.14)–(5.18) of II, but with M there replaced by M_1 and $\sigma \equiv 0$. These functions are linear in f_0 and g_0. The functions F'_{114} and G'_{114} are given by

$$F'_{114} = -\tfrac{1}{4}f_0 M_2 Df_{12}-\tfrac{1}{2}(Df_0)M_2 f_{12}+\tfrac{1}{2}f_{12}M_1 Df_0+\tfrac{1}{4}(Df_{12})M_1 f_0 \\ -\lambda^2 cT_0 g_0(g_{10}+\tfrac{1}{2}g_{12}), \quad (4.19)$$

$$G'_{114} = f_0 Dg_{10}+\tfrac{1}{2}f_0 Dg_{12}+\tfrac{1}{2}f_{12}Dg_0+g_{12}Df_0+\tfrac{1}{4}g_0 Df_{12}. \quad (4.20)$$

The boundary conditions on (4.17) and (4.18) are

$$u_{21} = Du_{21} = v_{21} = 0, \quad x = \pm\tfrac{1}{2}. \quad (4.21)$$

The homogeneous problem corresponding to (4.17), (4.18) and (4.21) is the eigenvalue problem at $O(\epsilon^{\frac{1}{2}})$, whose solution is (4.1). Therefore, in order for the differential system (4.17), (4.18) and (4.21) to have a solution it is necessary for the following orthogonality condition to be satisfied: if (f_0^+, g_0^+) is the adjoint function pair defined in II, equations (5.7)–(5.9), then we must multiply the right-hand sides of (4.17) and (4.18) by f_0^+ and g_0^+ respectively, add and integrate over $(-\frac{1}{2},\frac{1}{2})$ and require the resulting expression to be zero. Since ϕ derivatives do not occur on the left-hand sides of (4.17) and (4.18), ϕ is merely a parameter in the integration, so that the orthogonality condition yields a differential equation for $B(\phi)$. This is a nonlinear generalization of the linear amplitude equation (5.19) of II, and can be written as

$$kT_0^{\frac{1}{2}}\,dB/d\phi = \Gamma B(\phi)\cos\phi+(T_1\Gamma_3+\sigma_0\Gamma_5)\,B(\phi)-\Gamma'_4 B^3(\phi), \quad (4.22)$$

where

$$\left.\begin{aligned} \Gamma_0 &= \int_{-\frac{1}{2}}^{\frac{1}{2}}(F_{112}f_0^+ + G_{112}g_0^+)\,dx, \\ \Gamma &= -\Gamma_0^{-1}\int_{-\frac{1}{2}}^{\frac{1}{2}}(F_{111}f_0^+ + G_{111}g_0^+)\,dx, \\ \Gamma_3 &= -\Gamma_0^{-1}\int_{-\frac{1}{2}}^{\frac{1}{2}}f_0^+F_{113}\,dx, \\ \Gamma'_4 &= \Gamma_0^{-1}\int_{-\frac{1}{2}}^{\frac{1}{2}}(F'_{114}f_0^+ + G'_{114}g_0^+)\,dx, \\ \Gamma_5 &= \Gamma_0^{-1}\int_{-\frac{1}{2}}^{\frac{1}{2}}[\tfrac{1}{2}\lambda^2 cT_0 g_0 f_0^+(x^2-\tfrac{1}{4})-xf_0 g_0^+]\,dx. \end{aligned}\right\} \quad (4.23)$$

The amplitude equation, being of Bernoulli type, is easily integrated to yield

$$B^2(\phi) = \frac{k}{\omega_4'} \frac{[E(2\pi)-1]E(\phi)}{\int_0^{2\pi} E(\phi)\,d\phi + [E(2\pi)-1]\int_0^{\phi} E(\phi)\,d\phi}, \tag{4.24}$$

where

$$E(\phi) \equiv \exp\left[\frac{\omega}{k}\sin\phi + \frac{1}{k}(T_1\omega_3 + \sigma_0\omega_5)\phi\right], \tag{4.25}$$

$$(\omega, \omega_3, \omega_4', \omega_5) \equiv 2T_0^{-\frac{1}{2}}(\Gamma, \Gamma_3, \Gamma_4', \Gamma_5). \tag{4.26}$$

The integration constant has been chosen so that $B^2(\phi)$ is periodic with period 2π in ϕ, so as to ensure that the velocity and pressure fields (3.6), (3.7), (3.8), (4.2) and (5.5) are single-valued in ϕ. We note for future reference that

$$\int_0^{2\pi} B^2(\phi)\,d\phi = 2\pi T_1\omega_3/\omega_4, \tag{4.26a}$$

where σ_0 has been eliminated by (4.10) and ω_4 is defined by (4.27) and (4.28).

Having found an expression for $B(\phi)$ in (4.24), we can now determine σ_0 by using (4.10). This gives

$$\sigma_0 = 6Q_0\Gamma_3T_1/\Gamma_4, \tag{4.27}$$

where

$$\Gamma_4 = \Gamma_4' - 6Q_0\Gamma_5.$$

As a consequence of (4.27), the combination $T_1\omega_3 + \sigma_0\omega_5$, which appears in (4.25), can be written as

$$\left.\begin{aligned} T_1\omega_3 + \sigma_0\omega_5 &= T_1\omega_3\omega_4'/\omega_4, \\ \text{where} \qquad \omega_4 &= 2T_0^{-\frac{1}{2}}\Gamma_4. \end{aligned}\right\} \tag{4.28}$$

With these relations (4.25) becomes

$$E(\phi) = \exp[(\omega/k)\sin\phi + (T_1\omega_3\omega_4'/k\omega_4)\phi]. \tag{4.29}$$

Subject to the determination of the constants Q_0, Γ, Γ_0, Γ_3, Γ_4', Γ_5, Γ_4 and the solution of the associated differential systems, we have now obtained the velocity field to order ϵ. This gives the nonlinear extension of the zeroth-order results (5.45)–(5.50) of II, which correspond to $O(\epsilon^{\frac{1}{2}})$ here. For the pressure we have calculated the dominant, $O(\epsilon^{-2})$, 'lubrication' pressure, and have also obtained the less-important 'Taylor-vortex' pressure to $O(\epsilon)$.

Before going on to discuss the details of the velocity and pressure fields, it is convenient to give the values of constants defined in (4.9) and (4.23). Much of the information can be obtained from Davey's classic (1962) paper, by relating the functions occurring here to those occurring there. Other constants have been obtained by integration of data supplied by Dr Davey, in the form of unpublished tables and private notes. The results we have are for the case in which the outer cylinder is fixed ($q_2 = 0$, $c = 2$), and are as follows:

$$\left.\begin{array}{lll} \lambda = 3{\cdot}13, & T_0 = 1695, & Q_0 = -0{\cdot}1365, \\ \Gamma = 23{\cdot}09, & \Gamma_0 = 16{\cdot}90, & \Gamma_3 = 0{\cdot}007356, \\ \Gamma_4 = 38{\cdot}27, & \Gamma_5 = -2{\cdot}38, & \Gamma_4' = 40{\cdot}22, \\ \omega = 1{\cdot}122, & \omega_4 = 1{\cdot}859, & \omega_4' = 1{\cdot}954, \\ \omega_3 = 0{\cdot}0003573. & & \end{array}\right\} \tag{4.30}$$

The meaning of the prime introduced in several parameters and functions is that primed quantities (e.g. Γ_4') include a contribution due to the plane Poiseuille or 'lubrication' effect, whereas unprimed quantities do not (e.g. Γ_4). The value of Γ quoted in (4.30) is that of II, where the value of $\lambda = 3{\cdot}127$ is slightly different. One final point we wish to make is that T_1 is a free parameter, whose meaning is discussed in the next section.

5. The flow and pressure fields

We now have the perturbation velocity field of (3.6), (3.7) and (3.8) to $O(\epsilon)$:

$$u = -\epsilon^{-\frac{1}{2}}B(\phi)f_0(x)\cos\lambda\xi - \epsilon B^2(\phi)f_{12}(x)\cos 2\lambda\xi + O(\epsilon^{\frac{3}{2}}), \tag{5.1}$$

$$\begin{aligned} v = {} & \epsilon^{\frac{1}{2}}B(\phi)\,g_0(x)\cos\lambda\xi + \epsilon[B^2(\phi)\,g_{10}(x) + (3Q_0\Gamma_3T_1/\Gamma_4)\,(x^2-\tfrac{1}{4}) \\ & + B^2(\phi)\,g_{12}(x)\cos 2\lambda\xi] + O(\epsilon^{\frac{3}{2}}), \end{aligned} \tag{5.2}$$

$$w = \epsilon^{\frac{1}{2}}\lambda^{-1}B(\phi)\,Df_0(x)\sin\lambda\xi + \epsilon(2\lambda)^{-1}B^2(\phi)\,Df_{12}(x)\sin 2\lambda\xi + O(\epsilon^{\frac{3}{2}}). \tag{5.3}$$

The functions f and g occurring here are related to those of Davey's (1962) paper ($\eta \rightarrow 1, m = 0$) as follows:

$$\left.\begin{aligned} f_0(x) \equiv \overline{u}_1(D),\quad g_0(x) \equiv v_1(D),\quad f_{12}(x) \equiv -\tfrac{1}{2}\overline{u}_2(D),\quad g_{12}(x) \equiv -\tfrac{1}{2}\overline{v}_2(D), \\ g_{10}(x) = -\tfrac{1}{2}F_0(x) - 3Q_0(x^2-\tfrac{1}{4}) = -\tfrac{1}{2}F_1(D) - 3Q_0(x^2-\tfrac{1}{4}), \end{aligned}\right\} \tag{5.4}$$

where a D in parentheses denotes Davey's functions. These are given with derivatives in table B of his paper, and (in more detail) in tables 8, 9, 10 and 11, lodged with the Editor of *Journal of Fluid Mechanics*. The normalization used is $g_0(0) = 1$.

For the pressure perturbation (3.16) we have

$$\begin{aligned} p = {} & \frac{-3T_1^{\frac{1}{2}}\epsilon^{-2}}{2ck^3}Q_0\left[\frac{k}{\omega_4'}\ln\left\{1+\frac{(E(2\pi)-1)\displaystyle\int_0^{\phi}E(\phi)\,d\phi}{\displaystyle\int_0^{2\pi}E(\phi)\,d\phi}\right\} - \omega_3T_1\phi/\omega_4 + O(\epsilon)\right] \\ & - \epsilon^{\frac{1}{2}}\lambda^{-2}B(\phi)\,(D^2-\lambda^2)\,Df_0(x)\cos\lambda\xi + \epsilon[B^2(\phi)\,l_{10}(x) + (6Q_0\omega_3T_1/\omega_4)\,l_{100}(x) \\ & + B^2(\phi)\,l_{12}(x)\cos 2\lambda\xi] + O(\epsilon^{\frac{3}{2}}). \end{aligned} \tag{5.5}$$

Although the $O(\epsilon^{\frac{1}{2}})$ and $O(\epsilon)$ 'Taylor-vortex' contributions are given above, because the functions are known from (4.5), (4.14) and (4.15), these terms are actually of higher order than the term $O(\epsilon^{-1})$ in (3.16). The next important term after the $O(\epsilon^{-2})$ term is the one $O(\epsilon^{-1})$, involving $q_1(\phi)$, which has not been calculated. The $O(\epsilon^{-2})$ pressure perturbation is 'lubrication' dominated, and is a function of ϕ only.

In order to render the formulae (5.1)–(5.3) more informative, it is necessary to examine the properties of the function $B(\phi)$. First of all we note that (4.24) differs from the form (II, equation (5.21)) given in linearized theory, namely

$$B^2(\phi) = B_0^2(k)\exp\{(\omega/k)(\sin\phi - 1)\}, \tag{5.6}$$

where $B_0(k)$ is there an arbitrary constant. Moreover (4.24) contains two free parameters, k and T_1. The parameter k, which also appears in (5.6), relates to the geometrical constraint (2.2), while T_1 defines the amount by which the Taylor number exceeds the critical value, which is T_0 when the term $O(\epsilon^2)$ in (3.4) is neglected.

In our analysis we have let $\epsilon \to 0$ subject to k and T_1 being kept as fixed parameters. We are now in a position to consider the possibility of varying either or both of k and T_1 as a second limit. One of our objects is to know to what extent our present nonlinear predictions regarding the function $B(\phi)$ differ from those of II. Now, we expect to retrieve linear theory by allowing the Taylor number to approach the critical value, which is T_0 if terms $O(\epsilon^2)$ are ignored. Thus the case $T_1 = 0$ should give linear theory, and this suggests that we consider a second limit $T_1 \to 0$, with k fixed.

The limit $T_1 \to 0$, k fixed

By a straightforward expansion of (4.24), together with use of standard Bessel integrals, we obtain

$$\begin{aligned} B^2(\phi) = \frac{T_1\omega_3}{\omega_4}\frac{\exp[(\omega/k)\sin\phi]}{I_0(\omega/k)}\bigg[1+\frac{T_1\omega_3\omega_4'}{k\omega_4}\bigg(\pi+\phi-\frac{I(\omega/k)}{I_0(\omega/k)} \\ -\frac{1}{I_0(\omega/k)}\int_0^{\phi}\exp[(\omega/k\sin\phi]\,d\phi\bigg)+O(T_1^2)\bigg], \end{aligned} \tag{5.7}$$

where
$$I_0(x) = (2\pi)^{-1}\int_0^{2\pi}\exp(x\sin\phi)\,d\phi$$

is the modified Bessel function of order zero, and $I(x)$ is related to the modified Struve function $L_0(x)$ (Abramowitz & Stegun 1964, p. 498), and is given by

$$\begin{aligned} I(x) &= \frac{1}{2\pi}\int_0^{2\pi}\phi\exp(x\sin\phi)\,d\phi = \pi I_0(x)-\tfrac{1}{2}\pi L_0(x) \\ &= \pi I_0(x)-I_0(x)\int_0^x\frac{dx}{xI_0^2(x)}\int_0^x I_0(x)\,dx. \end{aligned} \tag{5.8}$$

Clearly the dominant term, $O(T_1)$, of (5.7) is of the same form as the linear solution (5.6), but has the arbitrary constant $B_0^2(k)$ replaced by

$$B_0^2(k, T_1) = \frac{T_1\omega_3 e^{\omega/k}}{\omega_4 I_0(\omega/k)}. \tag{5.9}$$

In contrast to the result (5.6) of the linear problem, this formula has a definite value for prescribed k and $T_1 \geqslant 0$, corresponding to a definite geometry and Taylor number (3.4).

We can now follow $T_1 \to 0$ by a third limit, namely on k. If $T_1 \to 0$ followed by $k \to \infty$, then

$$B_0^2(k, T_1) \sim \frac{T_1\omega_3}{\omega_4}\left[1+\frac{\omega}{k}+\left(\frac{\omega}{2k}\right)^2+O(k^{-3})\right]. \tag{5.10}$$

Alternatively, if $T_1 \to 0$ followed by $k \to 0$, then

$$B_0^2(k, T_1) \sim \frac{T_1\omega_3}{\omega_4}\left(\frac{2\pi\omega}{k}\right)^{\frac{1}{2}}[1+O(k)]. \tag{5.11}$$

These formulae confirm the remarks made in II about the importance of nonlinearity in determining the form of $B_0(k)$, and indicate that the limit $T_1 \to 0$ corresponds to an approach to the linearized solution near to the critical Taylor number, but with a definite amplitude.

It is also instructive to consider the alternative (second) limit, $T_1 \to \infty$. As we shall see, this places Stuart's (1958) and Davey's (1962) nonlinear solutions for the concentric case in the context of the present calculations.

The limit $T_1 \to \infty$, k fixed

By an asymptotic expansion for large T_1, we obtain

$$B^2(\phi) \sim \frac{T_1\omega_3}{\omega_4} + \frac{\omega}{\omega_4'}\cos\phi + \frac{k\omega\,\omega_4}{T_1\omega_3\omega_4'^2}\sin\phi - \frac{1}{T_1^2}\left\{\frac{k^2\omega\omega_4^2}{\omega_3^2\omega_4'^3}\cos\phi + \frac{k\omega^2\omega_4^2}{2\omega_3^2\omega_4'^3}\sin 2\phi\right\} + O(T_1^{-3}). \quad (5.12)$$

The most noticeable feature of this formula is that $B(\phi)$ becomes uniform in ϕ when $T_1 \to \infty$. In this limit, in which only the leading term of (5.12) is retained, the velocity field (5.1)–(5.3) is independent of ϕ. With use of (4.19), (4.20), (4.23), (4.28) and (5.4), it can be seen to be identical with the solution of Davey (1962). The leading term of (5.12) can also be obtained from the amplitude equation (4.22). The balance is between the last two terms,

$$(T_1\Gamma_3 + \sigma_0\Gamma_5)B(\phi) = \Gamma_4' B^3(\phi),$$

which are of order T_1; the other terms are negligible. Moreover, although the eccentricity ϵ remains in the perturbation velocity field (3.6)–(3.8) even when $T_1 \to \infty$, it does so only in the form ϵT_1, which, by (3.4), is $T - T_0$ to the order of our present calculation. The parameter k disappears from (5.12), and therefore from the velocity field. Finally, using the leading term of (5.12) for $B^2(\phi)$, together with (4.7) and (4.27), we can see that the lubrication pressure gradient $q_0(\phi)$ is zero in the limit $T_1 \to \infty$.

We now summarize our knowledge of the role of the parameter T_1, which we consider to lie in the range $0 \leqslant T_1 < \infty$. The lower end of the range yields a nonlinear modification to the results of linear theory for the eccentric case, and the upper end an approach to the equilibrium finite amplitude concentric solution. In the latter case, the combination $\epsilon T_1 = T - T_0$ determines the magnitude of the supercritical Taylor number T, but subject to the over-riding first limit $\epsilon \to 0$.

The position of maximum vortex activity

Following II we consider one possibly relevant measure of the strength of vortex activity, which may correspond with observation, to be the radial gradient of the axial velocity evaluated at the outer cylinder.

From (5.3) we see that the dominant term, $O(\epsilon^{\frac{1}{2}})$, is largest at $\lambda\xi = \frac{1}{2}\pi$, where the second term, $O(\epsilon)$, is zero. It seems reasonable, therefore, to confine our attention to the dominant term, so that we need only to consider the azimuthal variation of the functions $B(\phi)$. (In contrast, II takes the *linear* problem to a higher order, and thus a correction is obtained to the above leading term. This would correspond to $O(\epsilon^{\frac{3}{2}})$ here, but we have not calculated it.) Moreover in the present approximation, where only the leading term is kept, ϕ is equivalent to the actual polar angle on the outer cylinder, as discussed in II. The outer cylinder is, of course, especially relevant in the context of observations made from outside the cylinder, in that flow conditions near the outer cylinder are most noticeable.

Equations (4.24), (4.25) and (4.29) can be used to show that $B(\phi)$ has its maximum at a location ϕ_M given by

$$k[E(2\pi)-1]\,E(\phi_M) = (\omega\cos\phi_M + T_1\omega_3\omega_4'/\omega_4) \times\left\{\int_0^{2\pi} E(\phi)\,d\phi + [E(2\pi)-1]\int_0^{\phi_M} E(\phi)\,d\phi\right\}. \tag{5.13}$$

Either from this relation, or by differentiation of (5.7) and (5.12), it is possible to obtain the value of ϕ_M for which $B(\phi)$ is a maximum in the limits $T_1 \to 0$ and $T_1 \to \infty$ respectively.

For $T_1 \to 0$ we obtain

$$\phi_M = \frac{\pi}{2} + T_1\frac{\omega_3\omega_4'}{\omega\omega_4}\left[1-\frac{e^{\omega/k}}{I_0(\omega/k)}\right] + O(T_1^2), \tag{5.14}$$

and for $T_1 \to \infty$

$$\phi_M = \frac{k\omega_4}{T_1\omega_3\omega_4'} - \frac{k\omega}{2T_1^2}\left(\frac{\omega_4}{\omega_3\omega_4'}\right)^2 + O(T_1^{-3}). \tag{5.15}$$

The first of these results corresponds to a perturbation away from linearized theory. The second, in which limit we approach the concentric case, gives $\phi_M \to 0$, though this has no real significance because the flow becomes axisymmetric. Of greater importance is that (5.15) gives a positive value for ϕ_M when T_1 is large. The above results suggest that, as T_1 ranges from 0 to ∞, the position of maximum vortex activity varies from $\phi = \frac{1}{2}\pi$ down to $\phi = 0$. This is confirmed by direct numerical calculation of $B^2(\phi)$ for a variety of cases.

It is very interesting to compare the position of maximum vortex strength with the experimental observations of Vohr (1968). For his experiments with $\delta = 0{\cdot}099$ and $\epsilon = 0{\cdot}475$ we find that $k = 0{\cdot}31$. For this value of k the numerical variation of ϕ_M with T_1 is given in table 1. Vohr has informed us that his observation, namely that the maximum vortex strength occurred at about 50° downstream of the position of widest gap, was for a speed about 1·1 times the critical speed. In terms of Taylor numbers this would be about 20% above critical, which means that $T - T_0 \simeq 0{\cdot}20 \times 1695 = 340$. For $\epsilon = 0{\cdot}475$ this gives $T_1 \simeq 700$. From table 1 we find that ϕ_M is approximately 47° for $T_1 = 700$. Too much significance should not be assigned to this agreement between theory and experiment. For $\delta = 0{\cdot}099$ and $\epsilon = 0{\cdot}475$ corrections associated (i) with the next term in the velocity field, (ii) with the $O(\epsilon^2)$ term in the critical Taylor number and (iii) with the difference between the bipolar angle ϕ and the polar angle Θ measured around the outer cylinder should be important. Indeed for $\delta = 0{\cdot}099$ and $\epsilon = 0{\cdot}475$ we find from equations (6.18) and (6.21) of II that $\phi = 50°$ corresponds to $\Theta = 76°$. Moreover (to quote II) we note that the observation of about 50° was certainly qualitative and subjective, as Dr Vohr has explained to us, and that flow

T_1	ϕ_M	$B^2(\phi_M)$	T_1	ϕ_M	$B^2(\phi_M)$
0	$\frac{1}{2}\pi$	0	700	0·82	0·526
20	1·55	0·018	800	0·74	0·580
50	1·51	0·044	900	0·66	0·628
100	1·45	0·088	1000	0·59	0·671
200	1·33	0·173	1200	0·46	0·745
300	1·22	0·255	1400	0·36	0·806
400	1·11	0·332	1600	0·29	0·858
500	1·01	0·403	1800	0·24	0·904
600	0·91	0·467	2000	0·20	0·947

TABLE 1. ϕ_M and $B^2(\phi_M)$ for $k = 0{\cdot}31$ and several values of T_1

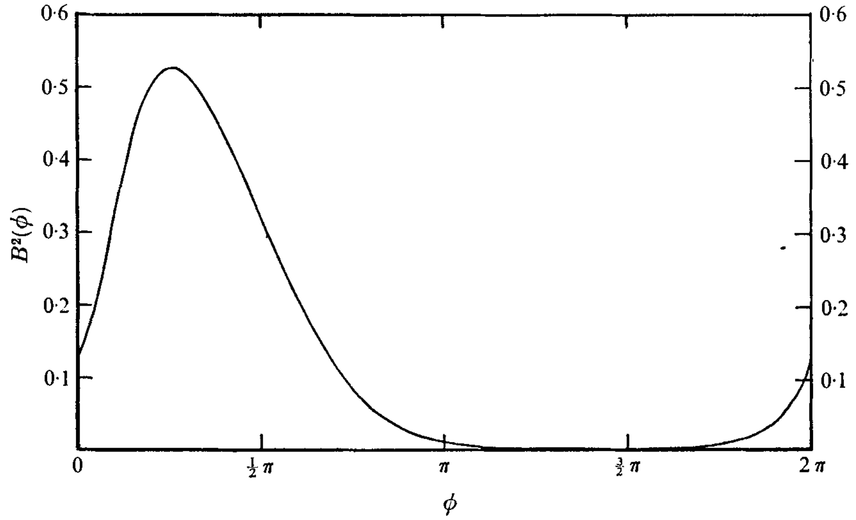

FIGURE 2. The variation of $B^2(\phi)$ with ϕ for $k = 0{\cdot}31$ and $T_1 = 700$.

reattachment, following separation, may have affected the observations. Nevertheless, it is clear that the present non-local stability analysis is capable of explaining a shift downstream in the position of maximum vortex strength from that of widest clearance as predicted by a local theory. The variation of $B^2(\phi)$ with ϕ for $k = 0{\cdot}31$ and $T_1 = 700$ is shown in figure 2. An experimental determination of the variation with Θ of the strength of the Taylor vortex would be helpful.

We conclude with a remark about the range of validity of the formulae (5.14) and (5.15) and the corresponding formulae (5.7) and (5.12) for $B^2(\phi)$. For $k = 0{\cdot}31$ a numerical calculation shows that the $O(T_1^2)$ term in (5.7) is 17 % of that $O(T_1)$ for $T_1 = 100$. Thus one might expect (5.7) and (5.14) to be valid for $T_1 < 100$ approximately. Similarly for $k = 0{\cdot}31$ a numerical calculation shows that the second term in (5.12) is 30 % of that $O(T_1)$ for $T_1 = 10^4$. This suggests that T_1 must be greater than 10^4 for (5.12) and (5.15) to be valid.

6. The torque and load

For laminar flow, in the absence of Taylor vortices, formulae for the torque and force on the inner cylinder are well known in the limit $\alpha \to 0$, $R_M \to 0$ (the lubrication approximation). Paper I gives the torque and load expressions by series expansion to order α and R_M for arbitrary ϵ in the range $0 \leqslant \epsilon < 1$. For the Taylor-vortex problem in the concentric case with the outer cylinder at rest ($q_2 = 0$), Stuart (1958), Davey (1962), Reynolds & Potter (1966) and Kirchgässner & Sorger (1969) have given expressions for the torque to varying degrees of accuracy by amplitude expansions. Here we extend the torque calculation to the eccentric case with $q_2 = 0$, although only to second-order in velocity amplitude, and we also evaluate the Taylor-vortex force on the inner cylinder due to eccentricity.

In order to calculate the torque and force on the inner, rotating, cylinder, when the outer one is at rest, we need to consider the stress $\tau_{\rho\rho}$ directed along the outward normal to the inner cylinder in figure 1, and the stress $\tau_{\rho\phi}$ tangential to the cylinder in the anticlockwise sense. Integration over ϕ and ξ must be performed. Since we have considered the cylinders to be infinitely long, corresponding to many Taylor-vortex wavelengths in an experiment, we require only the mean parts (with respect to ξ) of the velocity and pressure fields; the other Fourier components integrate out.

The normal and tangential stresses on a surface ρ = constant are given by

$$\tau_{\rho\rho} = -p'\sigma + \frac{2\mu}{a\rho}\left[\rho J^{\frac{1}{2}}\frac{\partial u_\rho}{\partial \rho} - u_\phi \frac{\partial J^{\frac{1}{2}}}{\partial \phi}\right], \tag{6.1}$$

$$\tau_{\rho\phi} = \frac{\mu}{a\rho}\left[\rho^2 \frac{\partial}{\partial \rho}\left(\frac{J^{\frac{1}{2}} u_\phi}{\rho}\right) + \frac{\partial}{\partial \phi}(J^{\frac{1}{2}} u_\rho)\right], \tag{6.2}$$

where σ denotes the density and μ the viscosity.

On the inner cylinder, which is represented by $\rho = 1$, the directions of the normal (6.1) and tangential (6.2) stresses are coincident with the radial and azimuthal directions in a polar co-ordinate system based on the centre of the inner cylinder. Thus the torque and the force components acting on a length l of the inner cylinder are given by

$$T_i = la^2 \int_0^{2\pi} (\tau_{\rho\phi})_{\rho=1} d\theta, \tag{6.3}$$

$$F_X = la \int_0^{2\pi} (\tau_{\rho\rho}\cos\theta - \tau_{\rho\phi}\sin\theta)_{\rho=1} d\theta, \tag{6.4}$$

$$F_Y = la \int_0^{2\pi} (\tau_{\rho\rho}\sin\theta + \tau_{\rho\phi}\cos\theta)_{\rho=1} d\theta, \tag{6.5}$$

where (X, Y) are Cartesian co-ordinates along and normal to $\theta = \phi = 0$ (figure 1), and the stesses are means with respect to ξ.

In order to evaluate the above formulae correctly, we need the relation between θ and ϕ since the stresses on the inner cylinder are given in terms of ϕ; this relation is given in I. For the Taylor-vortex correction in the present analysis, which is $O(\epsilon)$ for both torque and force, it is sufficient to let $\theta \equiv \phi$.

We first consider the Taylor-vortex contributions, which we denote by an overbar. The contributions from the basic flow, as given in I, will be added later.

From the analysis of this paper, with $q_2 = 0$ and $c = 2$, we find

$$\bar{\tau}_{\rho\rho} = -\frac{\nu^2\sigma}{a^2\alpha^2}\frac{T^{\frac{1}{2}}\epsilon^{-2}}{4ck^3}\int_0^\phi q_0(\phi)\,d\phi[1+O(\epsilon)]$$

$$= \frac{6Q_0\mu q_1\epsilon}{a\delta^2}\int_0^\phi\left[B^2(\phi)-\frac{\Gamma_3 T_1}{\Gamma^4}\right]d\phi[1+O(\epsilon)], \tag{6.6}$$

$$\bar{\tau}_{\rho\phi} = -\frac{\mu q_1\epsilon}{a\delta}\left[B^2(\phi)\{\tfrac{1}{2}DF_0(-\tfrac{1}{2})-3Q_0\}+\frac{3Q_0\Gamma_3 T_1}{\Gamma_4}\right][1+O(\epsilon)] \tag{6.7}$$

for the mean Taylor-vortex stresses on the inner cylinder. It is seen that the normal stress, which is dominated by the contribution of the 'lubrication' pressure field $q_0(\phi)$, is larger than the tangential stress by a factor δ^{-1}. Thus the force contributions attributable to the Taylor vortices come dominantly from (6.6) and thus from the 'lubrication' effect. Substituting (6.7) in (6.3) and (6.6) in (6.4) and (6.5), we find the following expressions for the torque and load on the inner cylinder, where we have also included contributions from the basic flow:

$$T_i = \frac{-2\pi\mu q_1 la}{\delta}\left\{[1+O(\epsilon^2)]+\frac{\epsilon T_1\Gamma_3}{2\Gamma_4}DF_0(-\tfrac{1}{2})[1+O(\epsilon)]\right\}, \tag{6.8}$$

$$F_X = \frac{-\pi\mu q_1 l}{\delta^2}\left\{[\tfrac{1}{5}R_m\epsilon(1+O(\epsilon^2))]+\frac{6Q_0\epsilon}{\pi}F_s(k,T_1)[1+O(\epsilon)]\right\}, \tag{6.9}$$

$$F_Y = \frac{-6\pi\mu q_1 l}{\delta^2}\left\{[\epsilon(1+\tfrac{1}{2}\delta+O(\epsilon^3))]-\frac{Q_0\epsilon}{\pi}F_c(k,T_1)[1+O(\epsilon)]\right\}. \tag{6.10}$$

In these formulae

$$F_c(k,T_1) = \int_0^{2\pi}B^2(\phi)\cos\phi\,d\phi, \quad F_s(k,T_1) = \int_0^{2\pi}B^2(\phi)\sin\phi\,d\phi. \tag{6.11}$$

The contributions from the basic flow, which are given first with the appropriate errors, are taken from equations (76)–(78) of I, but with (76) multiplied by an omitted ϵ and all three formulae subject to expansions valid for small ϵ. In the above formulae (6.8)–(6.10), relation (3.4) shows that ϵ may be replaced by $(T-T_0)/T_1$ and for later comparisons this is helpful.

Thus, using (3.4) in this way, we see that formula (6.8) for the torque is precisely that of Davey (1962) for the concentric case. However, it is worth noting that the Taylor-vortex contribution to the torque, being $O(\epsilon)$ for T_1 fixed, is larger than the correction $O(\epsilon^2)$ to the basic flow due to eccentricity. In formula (6.9), we note especially that the Taylor-vortex contribution to F_X, being $O(\epsilon)$, is *larger* in ϵ ordering than the contribution of the basic flow due to inertia, which is proportional to ϵR_M, where $R_M = (q_1 a/\nu)\alpha^2 = O(\epsilon)$. Moreover in (6.10) the Taylor-vortex contribution has larger order than the curvature (δ) correction due to the basic flow.

k	0·1		0·2		0·31		0·5	
T_1	F_c	F_s	F_c	F_s	F_c	F_s	F_c	F_s
20	0·0008	0·0230	0·0005	0·0219	0·0004	0·0205	0.0002	0·0177
50	0·0048	0·0574	0·0031	0·0546	0·0022	0·0511	0·0014	0·0443
100	0·0189	0·1133	0·0122	0·1085	0·0087	0·1018	0·0054	0·0884
200	0·0695	0·2169	0·0472	0·2120	0·0341	0·2006	0·0215	0·1752
300	0·1411	0·3067	0·1012	0·3068	0·0748	0·2938	0·0477	0·2592
400	0·2254	0·3824	0·1698	0·3907	0·1284	0·3795	0·0832	0·3388
500	0·3173	0·4449	0·2485	0·4632	0·1923	0·4564	0·1268	0·4133
600	0·4136	0·4953	0·3341	0·5242	0·2640	0·5239	0·1773	0·4817
700	0·5119	0·5347	0·4237	0·5745	0·3412	0·5819	0·2335	0·5437
800	0·6106	0·5640	0·5152	0·6147	0·4219	0·6306	0·2940	0·5988
900	0·7082	0·5842	0·6071	0·6457	0·5042	0·6704	0·3575	0·6471
1000	0·8039	0·5963	0·6981	0·6682	0·5869	0·7019	0·4231	0·6887
2000					1·2663	0·7108		

k	0·7		1·0		1·5		2·0	
T_1	F_c	F_s	F_c	F_s	F_c	F_s	F_c	F_s
20	0.0001	0·0150	0·0001	0·0118	0·0000	0·0085	0·0000	0·0065
50	0·0009	0·0375	0·0005	0·0295	0·0003	0·0211	0·0002	0·0163
100	0·0036	0·0748	0·0021	0·0589	0·0010	0·0422	0·0006	0·0326
200	0·0141	0·1487	0·0082	0·1173	0·0041	0·0844	0·0024	0·0651
300	0·0314	0·2210	0·0184	0·1750	0·0092	0·1262	0·0054	0·0975
400	0·0552	0·2909	0·0325	0·2315	0·0162	0·1676	0·0096	0·1297
500	0·0848	0·3577	0·0503	0·2866	0·0253	0·2084	0·0149	0·1617
600	0·1198	0·4209	0·0716	0·3399	0·0362	0·2486	0·0214	0·1933
700	0·1594	0·4800	0·0961	0·3911	0·0489	0·2880	0·0290	0·2246
800	0·2030	0·5347	0·1235	0·4400	0·0633	0·3265	0·0377	0·2554
900	0·2499	0·5848	0·1537	0·4863	0·0794	0·3639	0·0474	0·2858
1000	0·2994	0·6301	0·1861	0·5301	0·0970	0·4003	0·0582	0·3156

TABLE 2. F_c and F_s for various values of k and T_1

In order to have more knowledge of the force, we need to evaluate the integrals (6.11). From (4.24) and (5.7) we find, for $T_1 \rightarrow 0$, that

$$F_c(k,T_1) = \frac{2\pi T_1^2 \omega_3^2 \omega_4'}{\omega\omega_4^2}\left[-1+\frac{I_0(2\omega/k)}{I_0^2(\omega/k)}\right]+O(T_1^3), \tag{6.12}$$

$$F_s(k,T_1) = \frac{2\pi T_1 \omega_3}{\omega_4}\frac{I_1(\omega/k)}{I_0(\omega/k)}+O(T_1^3). \tag{6.13}$$

On the other hand, for $T_1 \rightarrow \infty$, formulae (4.24) and (5.12) yield

$$F_c(k,T_1) = \frac{\pi\omega}{\omega_4'} - \frac{\pi k^2 \omega\omega_4^2}{T_1^2 \omega_3^2 \omega_4'^3}+O(T_1^{-3}), \tag{6.14}$$

$$F_s(k,T_1) = \pi k \omega\omega_4/T_1 \omega_3 \omega_4'^2+O(T_1^{-3}). \tag{6.15}$$

Furthermore, table 2 gives the results of computations of F_c and F_s for a set of

$T-T_0$	T_1	F_{YB}	F_{YTV}	F_{XB}	$-F_{XTV}$	$\frac{(p_B+p_{TV})_{\max}}{(p_B)_{\max}}$
$\delta = 0{\cdot}01,\ \epsilon = 0{\cdot}1,\ k = 0{\cdot}499$						
84·75	848	0·6030	0·0085	0·0826	0·0162	1·058
169·5	1695	0·6030	0·0226	0·0845	0·0214	1·096
254·25	2543	0·6030	0·0328	0·0864	0·0200	1·102
339·0	3390	0·6030	0·0385	0·0883	0·0170	1·102
508·5	5085	0·6030	0·0434	0·0919	0·0122	1·101
847·5	8475	0·6030	0·0458	0·0987	0·0074	1·098
$\delta = 0{\cdot}01,\ \epsilon = 0{\cdot}2,\ k = 0{\cdot}247$						
84·75	424	1·2067	0·0087	0·1543	0·0212	1·038
169·5	848	1·2067	0·0269	0·1579	0·0336	1·079
254·25	1271	1·2067	0·0457	0·1615	0·0377	1·096
339·0	1695	1·2067	0·0615	0·1649	0·0366	1·097
508·5	2543	1·2067	0·0806	0·1717	0·0283	1·096
847·5	4238	1·2067	0·0907	0·1844	0·0161	1·094
$\delta = 0{\cdot}01,\ \epsilon = 0{\cdot}3,\ k = 0{\cdot}163$						
84·75	283	1·8147	0·0080	0·2046	0·0229	1·026
169·5	565	1·8147	0·0258	0·2095	0·0390	1·057
254·25	848	1·8147	0·0465	0·2142	0·0481	1·082
339·0	1130	1·8147	0·0669	0·2188	0·0516	1·089
508·5	1695	1·8147	0·1007	0·2277	0·0479	1·089
847·5	2825	1·8147	0·1321	0·2446	0·0284	1·088
$\delta = 0{\cdot}01,\ \epsilon = 0{\cdot}4,\ k = 0{\cdot}120$						
84·75	212	2·4368	0·0074	0·2231	0·0238	1·019
169·5	424	2·4368	0·0243	0·2284	0·0420	1·041
254·25	636	2·4368	0·0450	0·2335	0·0542	1·063
339·0	848	2·4368	0·0665	0·2285	0·0614	1·078
508·5	1271	2·4368	0·1069	0·2483	0·0645	1·081
847·5	2119	2·4368	0·1623	0·2667	0·0466	1·079
$\delta = 0{\cdot}01,\ \epsilon = 0{\cdot}5,\ k = 0{\cdot}093$						
84·75	170	3·0946	0·0069	0·1974	0·0243	1·014
169·5	339	3·0946	0·0231	0·2021	0·0439	1·030
254·25	509	3·0946	0·0431	0·2066	0·0583	1·046
339·0	678	3·0946	0·0648	0·2111	0·0681	1·060
508·5	1017	3·0946	0·1078	0·2197	0·0770	1·070
847·5	1695	3·0946	0·1781	0·2360	0·0666	1·069

TABLE 3. Continued on next page.

values of k and T_1. This forms a representative selection of results obtained by Dr P. A. McGloin and available from the authors. For a given geometry, k is given by (2.2) while for a given supercritical Taylor number T_1 is given by (3.4) with the term $O(\epsilon^2)$ ignored. Thus formulae (6.9) and (6.10), together with table 2 and any necessary extensions and interpolations, can be used to calculate the components of the force. In some circumstances formulae (6.12)–(6.15) may be useful.

Some typical values for the force components F_X and F_Y are given in table 3 for two values of δ, five values of ϵ and with the Taylor number at a level 5, 10, 15,

$T-T_0$	T_1	F_{YB}	F_{YTV}	F_{XB}	$-F_{XTV}$	$\frac{(p_B+p_{TV})_{\max}}{(p_B)_{\max}}$
			$\delta = 0{\cdot}1,\ \epsilon = 0{\cdot}1,\ k = 1{\cdot}577$			
84·75	848	0·6300	0·0017	0·2610	0·0086	1·018
169·5	1695	0·6300	0·0061	0·2672	0·0156	1·042
254·25	2543	0·6300	0·0119	0·2732	0·0201	1·066
339·0	3390	0·6300	0·0177	0·2791	0·0225	1·085
508·5	5085	0·6300	0·0272	0·2904	0·0230	1·107
847·5	8475	0·6300	0·0374	0·3120	0·0188	1·120
			$\delta = 0{\cdot}1,\ \epsilon = 0{\cdot}2,\ k = 0{\cdot}782$			
84·75	424	1·2608	0·0028	0·4876	0·0150	1·016
169·5	848	1·2608	0·0102	0·4991	0·0277	1·039
254·25	1271	1·2608	0·0203	0·5103	0·0368	1·063
339·0	1695	1·2608	0·0311	0·5213	0·0421	1·083
508·5	2543	1·2608	0·0503	0·5426	0·0449	1·104
847·5	4238	1·2608	0·0728	0·5828	0·0381	1·113
			$\delta = 0{\cdot}1,\ \epsilon = 0{\cdot}3,\ k = 0{\cdot}514$			
84·75	283	1·8959	0·0032	0·6465	0·0189	1·013
169·5	565	1·8959	0·0121	0·6617	0·0355	1·033
254·25	848	1·8959	0·0247	0·6766	0·0484	1·055
339·0	1130	1·8959	0·0389	0·6911	0·0572	1·073
508·5	1695	1·8959	0·0666	0·7193	0·0644	1·094
847·5	2825	1·8959	0·1040	0·7727	0·0579	1·102
			$\delta = 0{\cdot}1, \epsilon = 0{\cdot}4,\ k = 0{\cdot}378$			
84·75	212	2·5459	0·0034	0·7043	0·0212	1·010
169·5	424	2·5459	0·0127	0·7209	0·0402	1·026
254·25	636	2·5459	0·0264	0·7371	0·0558	1·044
339·0	848	2·5459	0·0424	0·7529	0·0675	1·060
508·5	1271	2·5459	0·0760	0·7837	0·0800	1·080
847·5	2119	2·5459	0·1288	0·8418	0·0777	1·088
			$\delta = 0{\cdot}1,\ \epsilon = 0{\cdot}5,\ k = 0{\cdot}293$			
84·75	170	3·2332	0·0034	0·6224	0·0225	1·008
169·5	339	3·2332	0·0128	0·6371	0·0431	1·020
254·25	509	3·2332	0·0268	0·6514	0·0606	1·033
339·0	678	3·2332	0·0437	0·6654	0·0744	1·046
508·5	1017	3·2332	0·0805	0·6926	0·0917	1·064
847·5	1695	3·2332	0·1461	0·7440	0·0961	1·072

TABLE 3. Loads for $\delta = 0{\cdot}01$ and $\delta = 0{\cdot}1$ and various values of k and T_1 (ϵ and $(T-T_0)$)

20, 30 or 50% above the critical value. For this purpose (6.9) and (6.10) are rewritten in the form

$$F_X = -\pi\mu q_1 l\,\delta^{-2}(F_{XB}+F_{XTV}), \tag{6.16}$$

$$F_Y = -\pi\mu q_1 l\,\delta^{-2}(F_{YB}+F_{YTV}). \tag{6.17}$$

The definitions of the Taylor-vortex contributions F_{XTV} and F_{YTV} follow from comparison of (6.16) and (6.17) with (6.19)–(6.11). On the other hand, the values of the basic-flow contributions F_{XB} and F_{YB} were not calculated from the forms (6.9) and (6.10) expanded in ϵ but from the full formulae (77) and (76) of I, with an omitted ϵ factor restored to the numerator of (76). It can be seen that, in the range of parameters given, F_{YTV} may be up to 8 % of F_{YB}. In contrast F_{XTV} may be as large as 25 % of F_{XB}, mainly because F_{XB} arises from an inertial laminar flow effect and is therefore proportional to R_m, and thus is smaller than F_{YB}. It should be remembered also that the present work is based upon the use of small parameters δ, R_m ($\sim (T\delta)^{\frac{1}{2}}$) and ϵ. It can be seen that some of the data of table 3 are on the borderline of violating those assumptions, especially for $\delta = 0{\cdot}1$, even if they have not actually violated them. The data for $\delta = 0{\cdot}01$ and for the smaller values of ϵ are more acceptable, since F_{XB} is small compared with F_{YB}, and the Taylor-vortex contributions are relatively small. It is interesting to note that, for the numerical calculations, $|F_{XTV}| < |F_{XB}|$, whereas the asymptotic form of (6.9) shows that F_{XTV} will dominate as $\epsilon \to 0$. Clearly, even $\epsilon = 0{\cdot}1$ is not small enough to counteract the effect of the numerical coefficients that appear in the formulae for F_{XB} and F_{XTV}. We note also that both F_{XB} and F_{XTV}, as well as F_{YTV}, are perturbations away from the main force component, F_{YB}.

Unfortunately we have not been able to find in the literature any experimental data with which to compare the computational results of table 3. However, Vohr (1967) has made measurements of the pressure distribution for $\delta = 0{\cdot}0104$ and for $\epsilon = 0{\cdot}20$, $0{\cdot}35$, $0{\cdot}51$ and $0{\cdot}68$, both for laminar flow and for Taylor-vortex flow. In one set of experiments the pressure dependence on angular position was measured and is shown in figures 19 and 20 of Vohr (1967). For the laminar case good agreement was found with the Sommerfeld pressure distribution ($\delta \to 0$, $R_m \to 0$, paper I), when an experimental technique to eliminate the inertial (R_m) effect was used. In the case of flow with Taylor vortices, the pressure distributions were found to be of similar shape to that of Sommerfeld, but of greater magnitude. The Taylor numbers are much too high for our theory, but we comment that, with

$$p' = (\nu q_1/a\delta^2)(p_B + p_{TV}), \tag{6.18}$$

we obtain values of p_{TV} comparable with experiment, or an order of magnitude lower.

In figure 24 of his paper, Vohr (1967) plots $(p_B + p_{TV})_{\max}/(p_B)_{\max}$. Values of this quantity are shown in table 3 (with $R_m \neq 0$ in the basic flow). For Vohr's cases of $\delta = 0{\cdot}0104$, $\epsilon = 0{\cdot}20$, $0{\cdot}35$ and $0{\cdot}51$ and with $R_m = 0$ in the basic flow, curves are shown in figure 3 of this quantity plotted against $R_\delta = q_1 a\delta/\nu$. The levelling off as R_δ increases represents the fact of this theory becoming invalid beyond $R_\delta \simeq 430$. Vohr's experiments show values comparable with those of figure 3 but at higher values of R_δ. The fact that our critical Taylor number is T_0, independent of ϵ to the present order of accuracy, may have some effect on the comparisons.

In addition, Vohr (1968) has made observations of the torque as a function of speed, with δ either $0{\cdot}01$ or $0{\cdot}1$ approximately and for several values of ϵ. The most noticeable feature is that the angle of bifurcation, between the torque curves for laminar and Taylor-vortex flows at the critical point for instability,

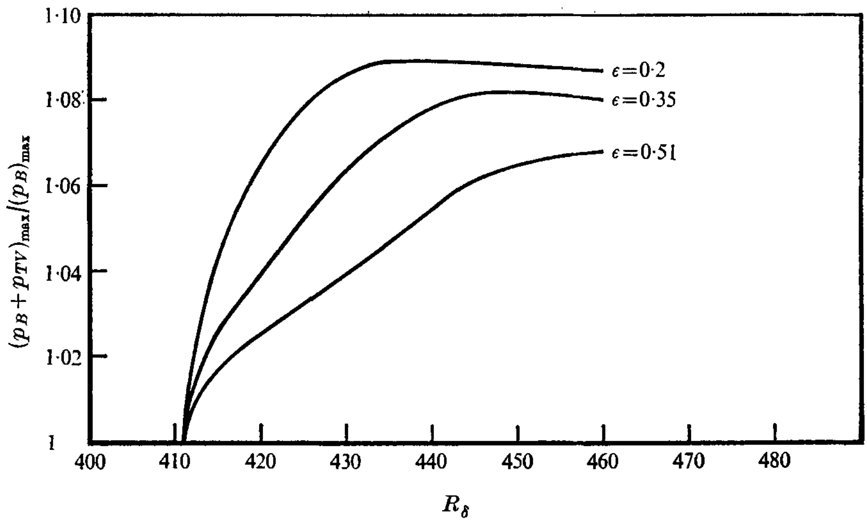

FIGURE 3. The variation of $(p_B+p_{TV})_{\max}/(p_B)_{\max}$ with R_δ for $\epsilon = 0{\cdot}2$, $0{\cdot}35$ and $0{\cdot}51$ and $\delta = 0{\cdot}0104$.

decreases as the eccentricity increases. Our torque formula, correct up to terms $O(\epsilon)$, is identical with that for the concentric case, and as a consequence cannot explain this.

7. Discussion

It is desirable to assess the three objectives of this paper, as stated in the introduction.

(i) A mathematical perturbation procedure has been devised in which the eccentricity tends to zero subject to two parameters k and T_1 being held fixed; the parameter k relates the clearance ratio and eccentricity, while T_1 determines the value of the supercritical Taylor number. For $T_1 \rightarrow 0$, as a second limit, the bifurcation point of linear theory is approached, but the arbitrary amplitude of II is now known; for $T_1 \rightarrow \infty$ the concentric nonlinear flow field is obtained.

(ii) It has been shown that the azimuthal position of maximum vortex activity varies with the value of T_1, being at a downstream location of 90° from the point of widest gap when $T_1 \rightarrow 0$ but at the point of widest gap when $T_1 \rightarrow \infty$. For Vohr's (1968) experiments with $\delta = 0{\cdot}099$ and $\epsilon = 0{\cdot}475$, and at a speed 10 % above critical, the position of maximum vortex activity is at an angle of about 50°. For our calculations we have $k = 0{\cdot}31$ and $T-T_0 = 0{\cdot}2T_0$, which implies that $T_1 = 700$, and we find a similar angle, although the values of eccentricity and clearance ratio are probably too large for the theory to be valid.

(iii) The torque and load formulae have been calculated to order ϵ, equivalent to an order of (velocity amplitude)^2 for the Taylor-vortex contributions. Of especial note is that the torque does not depend upon eccentricity at this order and is simply Davey's (1962) result. It is necessary to go to order ϵ^2 at least in order to remedy this feature, and to be able therefore to compare the results with Vohr's (1968) measurements of torque and its dependence on eccentricity. Unfortunately we have been unable to find any measurements of the dependence of load on the presence of Taylor vortices, but a limited, and somewhat unsatisfactory, comparison has been made with Vohr's (1967) measurements of pressure distributions. Of interest is our result (see (6.8)–(6.15) and table 3) that, for a range of T_1 upwards from zero, the main part of the Taylor-vortex load on the inner cylinder is in a direction along the line of centres, and therefore at right angles to the 'Reynolds' load. As the value of T_1 increases, the Taylor-vortex load vector swings towards the normal (Reynolds) direction, but naturally the analysis shows that the load disappears in the concentric limit ($T_1 \rightarrow \infty$).

We conclude this paper by calling attention to an error and several minor misprints in an earlier paper, *J. Fluid Mech.* vol. 54, 1972, p. 393.

(i) p. 405. The term $-(3D^2 - \lambda^2 - \frac{1}{2}\sigma) D^2 f_0$ in (5.31) should be

$$-(5D^2 - 3\lambda^2 - \tfrac{3}{2}\sigma) D^2 f_0.$$

As a consequence [p. 407, equation (6.3)] T_{21} should be 1904, and (p. 408) the factor $1 + 1{\cdot}125\epsilon^2$ in (6.14) should be $1 + 2{\cdot}624\epsilon^2$. The upshot of these changes is that the curves in figures 2, 3 and 4 are shifted upwards, agreeing better generally with the experiments. The authors very much regret this serious error. Corrections of less serious 'slips of the pen' follow.

(ii) p. 397. In (2.15) delete δ in $\delta\alpha\Psi_{01}$.

(iii) p. 398. In (3.8) in the expression for P replace σ by ϕ and ν by ν^2 in the coefficient of p'.

(iv) p. 399. In (3.12), multiply the term $\dfrac{2\alpha^2\epsilon}{c}\dfrac{\partial}{\partial\phi}(\rho^{-1}J^{\frac{1}{2}})\,u_1$ by U.

(v) p. 399. In (3.15), replace $1 + \alpha x$ by ρ.

(vi) p. 402. In (4.16), replace $1 - \frac{1}{20}c^2$ by $1 - \frac{1}{12}c^2$.

(vii) p. 404. In (5.25), multiply $\lambda^2\, T_0 g_1$ by V_0.

(viii) p. 405. In (5.38) replace $(6x/x) f_1$ by $(6x/c) f_1$.

(ix) p. 412. In line 12 up replace $k = 0{\cdot}33$ by $k = 0{\cdot}31$.

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